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Characteristics of Invariant Curves of Plane Orbits

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The general characteristics of the totality of orbits in a two-dimensional potential, for a fixed value of energy, can be found by studying the invariant curves on a surface of section. The rotation number on each invariant curve is found as a function of its distance from the central invariant point, representing a stable periodic orbit. The tube orbits are represented by invariant curves of special forms which are called islands. There are no islands or tube orbits in separable potentials, while in the general case of non-integrable systems it seems that there are infinite sets of islands and tube orbits. The rotation curve (rotation number versus distance from the center) can be found approximately by means of the third integral. The rotation number near a stable symmetric periodic orbit is equal to $\pm\alpha T/2\pi$, where $\pm i\alpha$ are the two nonzero characteristic exponents and T the period of the periodic orbit. The stability of the central periodic orbit (near the y axis) and the orbit $y=0$ was studied. It was found that in some cases the orbits remain stable even when the zero velocity curves are open. Some applications to galactic problems are mentioned.

I. ROTATION CURVE

In this paper we deal with sets of orbits in a three-dimensional subspace of the phase space of a two-dimensional potential corresponding to fixed values of the energy. By studying the intersections of the orbits by a surface of section we can distinguish three types of orbits: (a) The isolating orbits, whose points of intersection lie on a smooth curve; such a curve is called an invariant curve. (b) The quasi-isolating (or semi-ergodic) orbits, whose points of intersection seem scattered at random, filling part of the available space (the space inside a limiting curve defined by the energy integral). A limiting case is the ergodic orbits whose points of intersection fill the whole available space. (c) The escaping orbits. Such orbits appear only if the energy integral surface extends to infinity.

A set of orbits in a given potential for a given energy constant is called isolating if all the orbits are isolating. It is called quasi-isolating if part of the orbits are isolating and part of them quasi-isolating. Finally, it is called ergodic if all the orbits are ergodic. The isolating and the ergodic cases seem to be rather exceptional, while the quasi-isolating case is the most general one. In the quasi-isolating cases the isolating orbits form a set of measure greater than zero. Exact theorems about the existence of sets of isolating orbits and of invariant curves in the immediate neighborhood of invariant points of the stable type are given by Moser (1962, 1967).

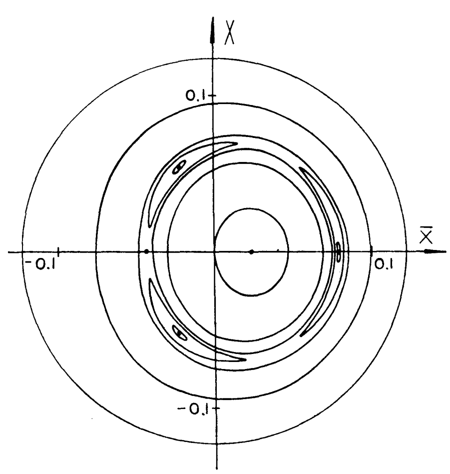

Fig. 1. Regular invariant curves and islands in the case $A=1.6$, $B=0.9$, $\epsilon=2$, $h=0.00765$.

We presently consider cases that are isolating or quasi-isolating, but nearly isolating, i.e., the area covered by the quasi-isolating orbits is small. In such cases the stable periodic orbits are represented by invariant points, surrounded by closed invariant curves.

On each invariant curve one can define a rotation number, which is the asymptotic value of the angle between two successive points of intersection, as seen from the invariant point. By "asymptotic" we mean the mean value of n angles between successive vectors, with the central invariant point as the origin, when n tends to infinitely. We use the circumference as the unit of angle. A rotation curve gives the rotation number along each invariant curve as a function of the distance from the "center" along a certain radius.

In many dynamical problems the potential is of the form

$$V=\tfrac{1}{2}(Ax^2+By^2)+\text{higher-order terms.} \qquad (1)$$

We usually take the plane $y=0$ of the space xyX as the surface of section. If the energy is not very large, there is a stable invariant point near the origin whenever the ratio $A^{\frac{1}{2}}/B^{\frac{1}{2}}$ is not near 1 or 2.

As an example we consider the case

$$V=\tfrac{1}{2}(Ax^2+By^2)-\epsilon xy^2. \qquad (2)$$

In this case the central stable invariant point is on the $\bar{x}$ axis ($\bar{x}=A^{\frac{1}{2}}x$; Fig. 1). The corresponding rotation curve is given in Fig. 2, where the $\bar{x}$ axis represents the value of $\bar{x}=A^{\frac{1}{2}}x$ at the point of intersection of each invariant curve with the $\bar{x}$ axis. The central invariant point corresponds to the maximum r, while the minimum r corresponds in this case to the outermost invariant curve

$$\bar{x}^2+X^2=2h,$$

where h is the total energy; this curve corresponds to the periodic orbit $y=0$.

The rotation number takes, eventually, rational values. For example, in Fig. 2 it takes the value $\frac{2}{3}$. At this point one sees a straight line segment on the rotation curve. All the points of this segment correspond to invariant curves with the same rational rotation number. There are three sets of such invariant curves in Fig. 1, which are called islands. Each set contains an infinity of islands surrounding one of the three stable invariant points, corresponding to a stable resonant periodic orbit, which makes four oscillations along the x axis, while it makes three oscillations along the y axis. Any orbit whose initial conditions are near this resonant periodic orbit always remains near it and is called a tube orbit. The corresponding invariant curve is composed of three islands.

On the other side of the rotation curve at the point $\frac{2}{3}$ we have an invariant point corresponding to an unstable periodic orbit. The rotation curve near this point seems to be discontinuous. In fact orbits very near the unstable periodic orbit do not give invariant curves at all. They are quasi-isolating orbits, whose points of intersection by the surface of section do not lie on a simple curve but show a slight dissolution, filling a narrow but finite strip.

These phenomena are clearer when the perturbation is large. A more detailed discussion of the forms of the islands and the dissolution of the invariant curves was given elsewhere (Contopoulos 1967). One can see that less marked discontinuities also appear for other rational values of r. However, the set of points belonging to these discontinuous regions has a small measure. Therefore for most values of $\bar{x}$ the corresponding rotation number lies on a smooth curve.

In the present paper we show that phenomena corresponding to apparent discontinuities of the rotation curve or its derivative (islands and dissolution of invariant curves) appear only in nonseparable dynamical systems. In separable systems the rotation curve is completely smooth and no islands appear for rational values of the rotation number.

If a dynamical system of the form (1) is separable, then a change of variables can bring the Hamiltonian to the form

$$H=H(\Phi_{10}',\Phi_{20}'), \tag{3}$$

where

$$\Phi_{10}'=\tfrac{1}{2}(Ax'^2+X'^2),$$
$$\Phi_{20}'=\tfrac{1}{2}(By'^2+Y'^2). \tag{4}$$

Then the equations of motion are

$$\frac{dx'}{dt}=\frac{\partial H}{\partial X'}=X'\frac{\partial H}{\partial \Phi_{10}'},\quad \frac{dX'}{dt}=-\frac{\partial H}{\partial x'}=-Ax'\frac{\partial H}{\partial \Phi_{10}'},$$
$$\frac{dy'}{dt}=\frac{\partial H}{\partial y'}=Y'\frac{\partial H}{\partial \Phi_{20}'},\quad \frac{dY'}{dt}=-\frac{\partial H}{\partial y'}=-By'\frac{\partial H}{\partial \Phi_{20}'}, \tag{5}$$

We derive that Φ_{10}', Φ_{20}' are integrals of motion, and the solutions for x', X', y', Y' are

$$x'=\frac{(2\Phi_{10}')^{\frac{1}{2}}}{A^{\frac{1}{2}}}\sin\left[A^{\frac{1}{2}}\frac{\partial H}{\partial \Phi_{10}'}(t-t_1)\right],$$
$$X'=(2\Phi_{10}')^{\frac{1}{2}}\cos\left[A^{\frac{1}{2}}\frac{\partial H}{\partial \Phi_{10}'}(t-t_1)\right],$$
$$y'=\frac{(2\Phi_{20}')^{\frac{1}{2}}}{B^{\frac{1}{2}}}\sin\left[B^{\frac{1}{2}}\frac{\partial H}{\partial \Phi_{20}'}(t-t_2)\right],$$
$$Y'=(2\Phi_{20}')^{\frac{1}{2}}\cos\left[B^{\frac{1}{2}}\frac{\partial H}{\partial \Phi_{20}'}(t-t_2)\right]. \tag{6}$$

If we write $\bar{x}'=A^{\frac{1}{2}}x'$ and take as the surface of section the plane $y'=0$ in the space $\bar{x}'y'X'$ we find that the moving point crosses this plane, moving upward, at the times

$$t=t_2+\frac{2k\pi}{B^{\frac{1}{2}}(\partial H/\partial\Phi_{20}')}, \tag{7}$$

where k is an integer. Then

$$\frac{\bar{x}'}{X'}=(2\Phi_{10}')^{\frac{1}{2}}\begin{matrix}\sin\\ \cos\end{matrix}\left\{A^{\frac{1}{2}}\frac{\partial H}{\partial\Phi_{10}'}\left[t_2-t_1+\frac{2k\pi}{B^{\frac{1}{2}}(\partial H/\partial\Phi_{20}')}\right]\right\}, \tag{8}$$

i.e., the invariant curves are circles $\bar{x}'^2+X'^2=2\Phi_{10}$. The angle of the direction of a point $(\bar{x}',X')$ from the origin with the $\bar{x}'$ axis is

$$\varphi=\frac{\pi}{2}-A^{\frac{1}{2}}\frac{\partial H}{\partial\Phi_{10}'}(t_2-t_1)-2k\pi\frac{A^{\frac{1}{2}}(\partial H/\partial\Phi_{10}')}{B^{\frac{1}{2}}(\partial H/\partial\Phi_{20}')}. \tag{9}$$

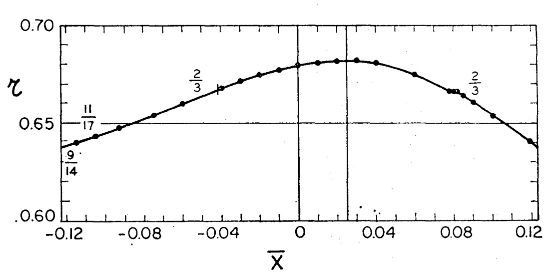

FIG. 2. Rotation curve in the case $A=1.6$, $B=0.9$, $\epsilon=2$, $h=0.00765$. The maximum rotation number at $\bar{x}=0.025$ is marked by a vertical line. The rational rotation numbers $r=n'/m'$ with $m'\leqslant 17$ are marked.

By subtracting the angles for two values, k and $k+1$, and dividing by 2π we find the rotation number

$$r=-\frac{A^{\frac12}(\partial H/\partial\Phi_{10}')}{B^{\frac12}(\partial H/\partial\Phi_{20}')}(\text{mod } 1). \qquad (10)$$

This rotation number is constant along each invariant curve.

If we set now

$$X'=0,\quad \Phi_{10}'=\tfrac12\bar{x}^2 \quad\text{and}\quad \Phi_{20}'=\tfrac12 Y'^2=h-\bar{x}^2, \qquad (11)$$

we find r as an analytic function of $\bar{x}$. If $H=\Phi_{10}'+\Phi_{20}'$ only, then $r=A^{\frac12}/B^{\frac12}=\text{const}$. In general, however, a separable Hamiltonian is of the form

$$H=\Phi_{10}'+\Phi_{20}'+\text{higher-order terms};$$

then

$$r=(A^{\frac12}/B^{\frac12})(1+\text{higher-order terms}), \qquad (12)$$

and r is variable. The rotation curve is, to a first approximation, a parabola, if the higher-order terms of H begin with quadratic terms in Φ_{10}', Φ_{20}'.

In all such cases the rotation curve is a smooth curve without any discontinuities or straight line segments, etc. In fact, r is a smooth function of $\bar{x}^2$ which does not behave differently for rational or irrational values of r.

If r is rational, the corresponding orbit is periodic. But orbits with initial positions on the same invariant curve are also periodic with the same period. Thus the invariant curves which correspond to a rational rotation number are composed only of invariable points, corresponding to an infinity of periodic orbits.

The characteristic exponents of these periodic orbits are equal to zero because of the existence of two analytic integrals of motion, namely, the Hamiltonian and the "third" integral Φ_{10}' (Poincaré 1892). However, these periodic orbits are unstable, because nearby orbits with a rotation number r slightly different from the given rational number n'/m' may go very far from the periodic orbits. For example, if the first invariant points of the two orbits are in the same direction, the nth invariant points will be at an angle $n(r-n'/m')$ apart.

The essential difference between separable and nonseparable systems is the nonexistence or existence of islands. This property is true for any surface of section because it corresponds to the nonexistence or existence of tube orbits and does not depend on the special surface of section used. It seems that in general two-dimensional systems are nonseparable; then the surfaces of section include islands of different orders.

There are some special nonseparable cases where we have a new analytic integral in addition to the Hamiltonian. We call such a case an integrable one. Such is, e.g., the case for the Hamiltonian

$$H=\tfrac12(Ax^2+X^2+By^2+Y^2)+\epsilon^3S_0,$$

where

$$S_0=2B^{\frac12}yY(X^3-3Ax^2X)-[3A^{\frac12}xX-(A^{\frac12}x)^3](Y^2-By^2)$$

when $3A^{\frac12}=2B^{\frac12}$. It is easily seen in this case that $d(\Phi_{10}+\Phi_{20})/dt=0$; hence

$$\Phi_{10}+\Phi_{20}=h_1=\text{const}$$

and

$$S_0=h_2=\text{const}.$$

If we intersect all the orbits by a surface of section $y=0$, we find

$$\bar{x}^2+X^2+Y^2=2h_1$$

and

$$(\bar{x}^3-3\bar{x}X^2)Y^2=h_2;$$

hence the equation of the invariant curves is

$$(\bar{x}^3-3\bar{x}X^2)(2h_1-\bar{x}^2-X^2)=h_2.$$

If we set

$$\bar{x}=r\cos\theta,\quad X=r\sin\theta,$$

we find

$$\cos 3\theta=h_2/r^3(2h_1-r^2).$$

If $h_2>0$ this equation represents three islands, symmetric with respect to the lines $\theta=0$, $\theta=2\pi/3$ and $\theta=4\pi/3$. If we set $\theta=0$ we find the intersections $\bar{x}_0$ of the islands by the $\bar{x}$ axis from the equation

$$\bar{x}_0^3(2h_1-\bar{x}_0^2)-h_2=0.$$

This equation has, in general, two positive roots smaller than $(2h)^{\frac12}$, on each side of $\bar{x}_0=(6h_1/5)^{\frac12}$. In the special case $\bar{x}_0=(6h_1/5)^{\frac12}$ we have a double root, which corresponds to a periodic orbit making three oscillations along the y axis and two along the x axis. This periodic orbit intersects the surface of section at the points $[r=(6h_1/5)^{\frac12};\ \theta=0,\ 2\pi/3,\ 4\pi/3]$, which are the centers of the three sets of islands.

For negative values of h_2 we have three more sets of islands around the points $[r=(6h_1/5)^{\frac12};\theta=\pi,\ 5\pi/3,\ \pi/3]$.

The two stable resonant periodic orbits are

$$\begin{aligned} x_1&=(6h_1/5A)^{\frac12}\sin[\omega A^{\frac12}(t-t_1)],\\ y_1&=(4h_1/5B)^{\frac12}\sin[\omega B^{\frac12}t],\\ X_1&=(6h_1/5)^{\frac12}\cos[\omega A^{\frac12}(t-t_1)],\\ Y_1&=(4h_1/5)^{\frac12}\cos[\omega B^{\frac12}t], \end{aligned}$$

where

$$\omega=1+2\epsilon^3(6h_1/5)^{\frac32}$$

and $\omega A^{\frac12}t_1=\frac12\pi$ (first orbit, $h_1>0$), or $-\frac12\pi$ (second orbit, $h_2<0$).

If we mark only every third intersection of an orbit by the plane $y=0$, we find one island for each orbit instead of three. We can study the rotation number along a set of islands with respect to the "central" invariant point. This new rotation number is a smooth function of the distance r' of an island from the "central" point along a fixed line. This is true because no islands of second order appear. In fact we know beforehand that all invariant curves are composed of three islands (except the unstable periodic orbit $\bar{x}=X=0$).

Therefore we conclude that integrable systems (i.e., systems having a second analytic integral besides the Hamiltonian) have a finite number of sets of islands on a surface of section. In the special case of separable systems we do not have any islands at all. On the other hand, nonintegrable systems probably have an infinity of sets of islands of different orders around the "central" invariant point, and also around the "center" of each island. The "center" of each set of islands corresponds to a stable resonant periodic orbit.

In nonintegrable systems each set of islands can be found by using a special form of the "third" integral (Contopoulos 1967). The "third" integral in this form, however, is not applicable, in general, for another set of islands. Therefore in integrable systems the "third" integral has a unique form, while in nonintegrable systems it probably has an infinity of forms.

In the case of large perturbations the rotation curve can no longer be drawn, except for small parts. This is due to the fact that for quasi-isolating (semi-ergodic) or ergodic orbits the rotation number is no longer defined; the mean value of n angles between successive points of intersection of an orbit (as seen from a fixed point) does not tend to a definite value as n tends to infinity. These phenomena and their explanation are described elsewhere (Contopoulos 1967).

We find presently the approximate form of the rotation curve in the case of the potential (2) when $A^{\frac{1}{2}}/B^{\frac{1}{2}}$ is not near 1 or 2.

For this purpose we use the von Zeipel method, as in a previous paper (Contoupoulos 1963). We use a generating function S to perform a change of variables, such that the new Hamiltonian depends only on the integrals Φ_{10}', Φ_{20}'. This is explicitly carried out up to terms of fourth degree in the variables.

The Hamiltonian

$$H=H_0+\epsilon H_1=\tfrac{1}{2}(Ax^2+X^2+By^2+Y^2)-\epsilon xy^2, \quad (13)$$

expressed in the new variables, becomes

$$H=H_0'+\epsilon^2 H_2'+\cdots, \quad (14)$$

where H_0' is of the same form as H_0; i.e.,

$$H_0'=\Phi_{10}'+\Phi_{20}', \quad (15)$$

and H_2' is quadratic in Φ_{10}', Φ_{20}'.

The new variables are found through the implicit relations

$$x'=\frac{\partial S}{\partial X'}, \quad y'=\frac{\partial S}{\partial Y'}, \quad X=\frac{\partial S}{\partial x}, \quad Y=\frac{\partial S}{\partial y}, \quad (16)$$

where

$$S=S_0+\epsilon S_1+\cdots. \quad (17)$$

Here

$$S_0=xX'+yY' \quad (18)$$

and the S_k are found as follows. By solving Eqs. (16) we find x, X, y, Y as series of x', X', y' Y', through the partial derivatives of S_k (considered as functions of the new variables). We introduce these values in H and gather together the terms of the same degree in ϵ. The terms of degree $k(k>1)$ are

$$X'\frac{\partial S_k}{\partial x'}-Ax'\frac{\partial S_k}{\partial X'}+Y'\frac{\partial S_k}{\partial Y'}-By'\frac{\partial S_k}{\partial Y'}+R_k=H_k', \quad (19)$$

where R_k depends only on the known terms S_0, S_1, $\cdots$, S_{k-1}.

If we write

$$H_k'-R_k=Q_k, \quad (20)$$

we find

$$S_k=\int Q_k dt, \quad (21)$$

where x', X', y', Y' in Q_k have been replaced by the solutions of the system

$$\frac{dx'}{X'}=\frac{dX'}{-Ax'}=\frac{dy'}{Y'}=\frac{dY'}{-By'}, \quad (22)$$

i.e.,

$$\begin{aligned} x'&=\frac{(2\Phi_{10}')}{A^{\frac{1}{2}}}\sin[A^{\frac{1}{2}}(t-t_1)], \quad y'=\frac{(2\Phi_{20}')^{\frac{1}{2}}}{B^{\frac{1}{2}}}\sin[B^{\frac{1}{2}}t] \\ X'&=(2\Phi_{10}')^{\frac{1}{2}}\cos[A^{\frac{1}{2}}(t-t_1)], \\ Y'&=(2\Phi_{20}')^{\frac{1}{2}}\cos[B^{\frac{1}{2}}t]. \end{aligned} \quad (23)$$

Q_k is a pure trigonometric function of t if H_k' is taken equal to the constant term of R_k (when R_k is expressed as a trigonometric function of t). Then S_k does not contain secular terms and can be expressed as a polynomial of degree $k+2$ in x', X', y', Y'.

It is easily found that $H_1'=0$, and

$$\begin{aligned} S_1=\int x'y'^2dt&=\frac{(2\Phi_{10}')^{\frac{1}{2}}(2\Phi_{20}')}{A^{\frac{1}{2}}B}\sin[A^{\frac{1}{2}}(t-t_1)]\sin^2[B^{\frac{1}{2}}t]dt=\frac{(2\Phi_{10}')^{\frac{1}{2}}(2\Phi_{20}')}{A^{\frac{1}{2}}B}\left\{-\frac{\cos[A^{\frac{1}{2}}(t-t_1)]}{2A^{\frac{1}{2}}}\right. \\ &\left.+\frac{2A^{\frac{1}{2}}\cos[A^{\frac{1}{2}}(t-t_1)]\cos[2B^{\frac{1}{2}}t]+4B^{\frac{1}{2}}\sin[A^{\frac{1}{2}}(t-t_1)]\sin[2B^{\frac{1}{2}}t]}{4(A-4B)}\right\} \\ &=-\frac{1}{A(A-4B)}[(2B-A)Xy^2+2XY^2+2AxyY]. \end{aligned} \quad (24)$$

Further

$$
\begin{aligned}
R_2={}&\frac{\partial H_1}{\partial X'}\frac{\partial S_1}{\partial x'}-\frac{\partial H_1}{\partial x'}\frac{\partial S_1}{\partial X'}+\frac{\partial H_1}{\partial Y'}\frac{\partial S_1}{\partial y'}-\frac{\partial H_1}{\partial y'}\frac{\partial S_1}{\partial Y'}-\frac{\partial H_0}{\partial X'}\left(\frac{\partial^2 S_1}{\partial x'^2}\frac{\partial S_1}{\partial X'}+\frac{\partial^2 S_1}{\partial x'\partial y'}\frac{\partial S_1}{\partial Y'}\right)-\frac{\partial H_0}{\partial Y'}\left(\frac{\partial^2 S_1}{\partial y'\partial x'}\frac{\partial S_1}{\partial X'}+\frac{\partial^2 S_1}{\partial y'^2}\frac{\partial S_1}{\partial Y'}\right)\\
&+\frac{\partial H_0}{\partial x'}\left(\frac{\partial^2 S_1}{\partial X'\partial x'}\frac{\partial S_1}{\partial X'}+\frac{\partial^2 S_1}{\partial X'\partial y'}\frac{\partial S_1}{\partial Y'}\right)+\frac{\partial H_0}{\partial y'}\left(\frac{\partial^2 S_1}{\partial Y'\partial x'}\frac{\partial S_1}{\partial X'}+\frac{\partial^2 S_1}{\partial Y'\partial y'}\frac{\partial S_1}{\partial Y'}\right)+\frac{1}{2}\frac{\partial^2 H_0}{\partial x'^2}\left(\frac{\partial S_1}{\partial X'}\right)^2+\frac{\partial^2 H_0}{\partial x'\partial y'}\frac{\partial S_1}{\partial X'}\frac{\partial S_1}{\partial Y'}\\
&+\frac{1}{2}\frac{\partial^2 H_0}{\partial y'^2}\left(\frac{\partial S_1}{\partial Y'}\right)^2-\frac{1}{2}\frac{\partial^2 H_0}{\partial x'\partial X'}\frac{\partial S_1}{\partial X'}\frac{\partial S_1}{\partial x'}-\frac{1}{2}\frac{\partial^2 H_0}{\partial x'\partial Y'}\frac{\partial S_1}{\partial X'}\frac{\partial S_1}{\partial y'}-\frac{1}{2}\frac{\partial^2 H_0}{\partial y'\partial X'}\frac{\partial S_1}{\partial Y'}\frac{\partial S_1}{\partial x'}-\frac{1}{2}\frac{\partial^2 H_0}{\partial y'\partial Y'}\frac{\partial S_1}{\partial Y'}\frac{\partial S_1}{\partial y'}\\
&+\frac{1}{2}\frac{\partial^2 H_0}{\partial X'^2}\left(\frac{\partial S_1}{\partial x'}\right)^2+\frac{\partial^2 H_0}{\partial X'\partial Y'}\frac{\partial S_1}{\partial x'}\frac{\partial S_1}{\partial y'}+\frac{1}{2}\frac{\partial^2 H_0}{\partial Y'^2}\left(\frac{\partial S_1}{\partial y'}\right)^2. \qquad (25)
\end{aligned}
$$

Using the value (24) for S_1 and the values (13) for H_0 and H_1 we find

$$
\begin{aligned}
R_2={}&\frac{1}{(A-4B)}\left\{\frac{y'^2}{A}[(2B-A)y'^2+2Y'^2]+\frac{4x'y'}{A}[2X'Y'+Ax'y']\right\}+\frac{1}{(A-4B)^2}\left\{-\frac{4X'Y'}{A}[2X'Y'+Ax'y']\right.\\
&-\frac{2Y'^2}{A}[(2B-A)y'^2+2Y'^2]+\frac{4(2B-A)X'Y'}{A^2}[2X'Y'+Ax'y']+\frac{4(2B-A)x'y'}{A}[2X'Y'+Ax'y']\\
&+\frac{2By'^2}{A}[(2B-A)y'^2+2Y^2]+\frac{4Bx'y'}{A}[2X'Y'+Ax'y']+\frac{1}{2A}[(2B-A)y'^2+2Y^2]^2+\frac{2B}{A^2}[2X'Y'+Ax'y']^2\\
&\left.+2y'^2Y'^2+\frac{2}{A^2}[(2B-A)X'y'+Ax'Y']^2\right\}. \qquad (26)
\end{aligned}
$$

The part of R_2 outside the trigonometric terms, when x', X', y', Y' are replaced by their values (23), is found after some operations as

$$H_2'=-\frac{\Phi_{20}'}{AB^2(A-4B)}[\tfrac{1}{4}(8B-3A)\Phi_{20}'+2B\Phi_{10}']. \qquad (27)$$

If we omit higher-order terms we have, to second-order approximation,

$$H'=\Phi_{10}'+\Phi_{20}'+[\epsilon^2\Phi_{20}'/AB^2(A-4B)]\times[\tfrac{1}{4}(8B-3A)\Phi_{20}'+2B\Phi_{10}']. \qquad (28)$$

Then using the formulas (6) we find two oscillations along the x' and y' axes with frequencies, to order ϵ^2,

$$\omega_1=A^{\frac{1}{2}}\{1+[\epsilon^2 2\Phi_{20}'/AB(A-4B)]\} \qquad (29)$$

and

$$\omega_2=B^{\frac{1}{2}}\{1+[\epsilon^2/AB^2(A-4B)]\times[\tfrac{1}{2}(8B-3A)\Phi_{20}'+2B\Phi'_{10}]\}. \qquad (30)$$

The rotation number is

$$r=\frac{\omega_1}{\omega_2}(\text{mod}1)=\frac{A^{\frac{1}{2}}}{B^{\frac{1}{2}}}\left\{1+\frac{\epsilon^2}{AB^2(A-4B)}\times[\tfrac{1}{2}(3A-4B)\Phi_{20}'-2B\Phi_{10}']\right\}(\text{mod } 1). \qquad (31)$$

If we write

$$\Phi_{20}'=h-\Phi_{10}'+O(\epsilon^2), \qquad (32)$$

where h is the total energy, we find

$$r=\frac{A^{\frac{1}{2}}}{B^{\frac{1}{2}}}\left\{1+\frac{\epsilon^2(3A-4B)h}{2AB^2(A-4B)}-\frac{3\epsilon^2\Phi_{10}'}{2B^2(A-4B)}\right\}(\text{mod } 1). \qquad (33)$$

This means that for $A>4B$ the rotation curve has a minimum at the center ($\Phi_{10}'=0$) and is concave upward, while for $A<4B$ the rotation curve has a maximum at the center and is convex upward.

The limiting values of r, to order ϵ^2, are

$$r=\frac{A^{\frac{1}{2}}}{B^{\frac{1}{2}}}\left[1+\frac{\epsilon^2(3A-4B)h}{2AB^2(A-4B)}\right](\text{mod } 1), \qquad (34)$$

and

$$r=\frac{A^{\frac{1}{2}}}{B^{\frac{1}{2}}}\left[1-\frac{2\epsilon^2 h}{AB(A-4B)}\right](\text{mod } 1), \qquad (35)$$

for $\Phi_{10}'=0$ and $\Phi_{10}'=h$, respectively. The last value corresponds to an orbit very near $y=0$. For example, for $A=1.6$, $B=0.9$, $h=0.00765$, we find $\Phi_{10}'=\frac{1}{2}\bar{x}'^2$

(for $X'=0$) and

$$r=-(4/3)\{1+\epsilon^2(-0.00177+0.463\bar{x}^2)\}\,(\mathrm{mod}\ 1)$$

and if $\epsilon=0$ we have $r=\frac{2}{3}$. For $\epsilon=2$ we find

$$r=\tfrac{2}{3}(1.014-3.70\bar{x}^2)$$

and the maximum value of r is 0.676 at $\bar{x}=0$.

The exact form of the rotation curve is given in Fig. 2. The maximum value is $r=0.682$ at $\bar{x}=0.025$.

In the case $A=0.4$, $B=0.9$, $h=0.00765$, we find

$$r=-(\omega_1/\omega_2)\,(\mathrm{mod}\ 1)=1-\tfrac{2}{3}[1+\epsilon^2(0.009+0.29\bar{x}^2)] \\ =\tfrac{1}{3}-0.006\epsilon^2-0.19\epsilon^2\bar{x}^2.$$

Therefore for $\epsilon\neq 0$ the rotation curve does not reach the rational value $\frac{1}{3}$. This behavior corresponds to the fact that when $A^{\frac{1}{2}}/B^{\frac{1}{2}}=\frac{2}{3}$ we do not have any resonance phenomena (resonant periodic orbits and tube orbits).

The above calculations are useful whenever ϵ is relatively small. They roughly give the range of variation of r and thus indicate the islands which should be looked for. If ϵ is large, inclusion of higher-order terms is quite necessary in order to find an approximately correct form of the rotation curve. Further, the above method does not give the straight segments of the rotation curve which represent the islands. These can be found only by using the special resonance forms of the third integral.

II. ROTATION NUMBERS AND CHARACTERISTIC EXPONENTS

The maximum (or minimum) rotation number corresponds to the central periodic orbit. We prove that this rotation number, which is defined as the limit of the rotation number of a nearby orbit, is equal to $\pm(\alpha T/2\pi)\,(\mathrm{mod}\ 1)$ where $\pm i\alpha$ are the two nonzero characteristic exponents of the stable periodic orbit and T its period.

Let

$$x=x(t),\quad y=y(t),\quad \dot{x}=X=X(t),\quad \dot{y}=Y=Y(t), \tag{36}$$

be a stable periodic orbit with period T, symmetric with respect to the x axis, and

$$x'(t)=x(t)+\xi_1,\qquad y'(t)=y(t)+\xi_2, \\ X'(t)=X(t)+\xi_3,\quad Y'(t)=Y(t)+\xi_4, \tag{37}$$

a nearby orbit corresponding to slightly different initial conditions. The ξ_i's are functions of the time and are considered to be small. Let $t=0$ when the periodic orbit crosses the x axis perpendicularly. Without loss of generality, we can assume that, at this time, a particle moving on the perturbed orbit (37) crosses the x axis as well. We call T_n the time when this particle crosses the x axis for the nth time in the same sense, and set

$$T_n=nT+\Delta T_n. \tag{38}$$

According to the above assumptions, we have

$$y'(T_n)=y(T_n)+\xi_2(T_n)=0 \tag{39}$$

for all $n>0$. Substituting T_n from Eq. (38) into Eq. (39), expanding in Taylor series and keeping only first-order terms in ΔT_n and ξ_i, we obtain

$$\Delta T_n=-\xi_2(nT)/Y(0), \tag{40}$$

because $y(nT)=y(0)=0$, and $Y(nT)=Y(0)$.

The functions ξ_i depend on the initial conditions of the perturbed orbit (37) and can be expressed, to first-order approximation, as a linear combination of four linearly independent solutions of the variational equations corresponding to the periodic orbit (36).

We find four solutions of the variational equations with initial conditions (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), and we construct a (4×4) matrix whose columns are these four solutions at $t=T$; this is called the monodromy matrix Δ. The equation $|\Delta-e^{\omega T}I|=0$, where I is a (4×4) unit matrix, gives the characteristic exponents ω. If the orbit (36) is stable we have two characteristic exponents zero and the other two conjugate pure imaginary $\pm i\alpha$. One can take as four independent solutions of the variational equations the vectors $f_1(t)$, $f_2(t)+tf_1(t)$, $f_3(t)\cos\alpha t-f_4(t)\sin\alpha t$ and $f_3(t)\sin\alpha t+f_4(t)\cos\alpha t$, with components $f_{1i}(t)$ $(i=1, 2, 3, 4)$ etc., where $f_1(0)$ and $f_3(0)\pm if_4(0)$ are the eigenvectors corresponding to the characteristic exponents zero and $\pm i\alpha$, and $f_2(0)$ is the solution of the system $(\Delta-I)f_2(0)=Tf_1(0)$ (Wintner 1947). Then the solution $\xi_i(t)$ can be expressed in the form

$$\xi_i(t)=r_1f_{1i}+r_2f_{2i}+tr_2f_{1i}+r_3(f_{3i}\cos\alpha t-f_{4i}\sin\alpha t) \\ +r_4(f_{3i}\sin\alpha t+f_{4i}\cos\alpha t)\quad (i=1,\cdots,4). \tag{41}$$

Here the f_{ij}'s are $O(1)$, and the r_i's are small constants, $O(\xi_i)$, depending on the initial conditions. Taking into account that at $t=0$, $y(0)=0$ and $X(0)=0$, we easily find that, in the case of the potential (2), we have $f_{11}(0)=f_{14}(0)=0$.

It can be shown (Hadjidemetriou 1967) that for $t=0$ we have

$$f_1(0)=\begin{bmatrix}0\\ b_1\\ b_2\\ 0\end{bmatrix},\quad f_2(0)=\begin{bmatrix}a_2\\ 0\\ 0\\ a_3\end{bmatrix},$$

$$f_3(0)=\begin{bmatrix}b_1\\ a_2\\ a_3\\ b_2\end{bmatrix},\quad f_4(0)=\begin{bmatrix}cb_1\\ -(1/c)a_2\\ -(1/c)a_3\\ cb_2\end{bmatrix}, \tag{42}$$

where

$$Y(0)=b_1,\quad \dot{X}(0)=b_2, \tag{43}$$

and a_2, a_3, c are constants, depending on the symmetric periodic orbit (36).

From Eqs. (41) and (42) we have, taking into account that $f_{ij}(nT)=f_{ij}(0)$, the relations

$$\xi_1(nT)=r_2a_2+b_1(r_3+cr_4)\cos n\alpha T + b_1(-cr_3+r_4)\sin n\alpha T, \quad (44)$$

$$\xi_2(nT)=r_1b_1+nTr_2b_1+\frac{a_2}{c}(cr_3-r_4)\cos n\alpha T + \frac{a_2}{c}(r_3+r_4c)\sin n\alpha T, \quad (45)$$

$$\xi_3(nT)=r_1b_2+nTr_2b_2+\frac{a_3}{c}(cr_3-r_4)\cos n\alpha T + \frac{a_3}{c}(r_3+r_4c)\sin n\alpha T. \quad (46)$$

Equation (40), because of Eqs. (43) and (45), becomes

$$\Delta T_n=-r_1-nTr_2-(a_2/cb_1)p_1\cos n\alpha T - (a_2/cb_1)p_2\sin n\alpha T, \quad (47)$$

where

$$p_1=cr_3-r_4, \quad p_2=r_3+cr_4. \quad (48)$$

The deviations in the coordinates x, X between the periodic and the perturbed orbits, when the latter crosses the x axis at the times $t=T_n (n>0)$, are given by

$$x'(T_n)-x(T_n)=\xi_1(T_n), \quad (49)$$

$$X'(T_n)-X(T_n)=\xi_3(T_n). \quad (50)$$

Expanding in Taylor series, we have, to first-order terms in the r_i's,

$$\delta x_n=x'(T_n)-x(0)=\xi_1(nT), \quad (51)$$

$$\delta X_n=X'(T_n)-X(0)=\xi_3(nT)+b_2\Delta T_n, \quad (52)$$

because $x(nT)=x(0)$, $X(nT)=X(0)=0$, $X(nT)=X(0)$, $\dot{X}(nT)=\dot{X}(0)=b_2$. Hence, using the values (44), (46), and (47), we find

$$\delta x_n=x'(T_n)-x(0) =r_2a_2+b_1[p_2\cos n\alpha T-p_1\sin n\alpha T], \quad (53)$$

$$\delta X_n=X'(T_n)-X(0)=(1/c)[a_3-a_2(b_2/b_1)] \times[p_1\cos n\alpha T+p_2\sin n\alpha T]. \quad (54)$$

The quantities δx_n, δX_n are the coordinates of the points of the invariant curve of the perturbed orbit (37) with respect to a frame of reference centered at the invariant point $[x(0),X(0)]$ of the periodic orbit (36). These quantities satisfy the relation

$$[(\delta x_n-r_2a_2)^2/A^{*2}]+[(\delta X_n)^2/B^{*2}]=1, \quad (55)$$

where

$$A^{*2}=b_1^2(p_1^2+p_2^2)=b_1^2(1+c^2)(r_3^2+r_4^2)$$

and (56)

$$B^{*2}=[(b_1a_3-a_2b_2)^2/b_1^2c^2](1+c^2)(r_3^2+r_4^2).$$

Hence, the invariant curves of the orbits in the neighborhood of the periodic orbit (36) are ellipses whose centers are at the points $[x(0)+r_2a_2, X(0)]$. We note that the centers of all ellipses lie on the x axis, since $X(0)=0$.

Let us call now θ_n the angle between the x axis and the line defined by the center of the ellipse (55) and the nth point on the invariant curve. Using Eqs. (53) and (54) we obtain

$$\tan\theta_n=\frac{X'(T_n)-X(0)}{x'(T_n)-x(0)+r_2a_2} =\sigma\frac{p_1\cos n\alpha T+p_2\sin n\alpha T}{p_2\cos n\alpha T-p_1\sin n\alpha T}, \quad (57)$$

where

$$\sigma=(a_3b_1-a_2b_2)/cb_1^2. \quad (58)$$

If we define the angle $\varphi(-\frac{1}{2}\pi<\varphi<\frac{1}{2}\pi)$ by the relation

$$\tan\varphi=p_1/p_2, \quad (59)$$

Eq. (57) becomes

$$\tan\theta_n=\sigma\tan(n\alpha T+\varphi). \quad (60)$$

From this equation we deduce that the rotation angle r, defined as the mean value of the angles $(\theta_n-\theta_{n-1})$, is equal to

$$r=\pm\alpha T/2\pi \pmod 1, \quad (61)$$

where the plus sign corresponds to $\sigma>0$ and the minus sign to $\sigma<0$. In fact if, say, $\sigma>0$ and $2k\pi<n\alpha T+\varphi<2k\pi+\frac{1}{2}\pi$, then $n\alpha T+\varphi<\theta_n<2k\pi+\frac{1}{2}\pi$. Hence

$$\alpha T<(\theta_n-\varphi)/n<\alpha T+(\pi/2n);$$

therefore

$$\lim(\theta_n-\varphi)/n=\alpha T.$$

As an example, we consider the potential field (2), for $\epsilon=0$. In this case, the general solution is

$$\begin{aligned} x&=x_0\cos A^{\frac{1}{2}}t+(X_0/A^{\frac{1}{2}})\sin A^{\frac{1}{2}}t,\\ y&=y_0\cos B^{\frac{1}{2}}t+(Y_0/B^{\frac{1}{2}})\sin B^{\frac{1}{2}}t,\\ X&=-x_0A^{\frac{1}{2}}\sin A^{\frac{1}{2}}t+X_0\cos A^{\frac{1}{2}}t,\\ Y&=-y_0B^{\frac{1}{2}}\sin B^{\frac{1}{2}}t+Y_0\cos B^{\frac{1}{2}}t, \end{aligned} \quad (62)$$

where the zero subscript denotes the initial conditions.

The characteristic exponents of a certain periodic orbit can be found from the trace of the monodromy matrix Δ. The four columns of this matrix are the partial derivatives of the functions x, y, X, Y of the periodic orbit with respect to x_0, y_0, X_0, Y_0 respectively, for $t=T$. Hence, using Eqs. (62), we obtain

$$\Delta=\begin{pmatrix} \cos A^{\frac{1}{2}}T & 0 & (1/A^{\frac{1}{2}})\sin A^{\frac{1}{2}}T & 0 \\ 0 & \cos B^{\frac{1}{2}}T & 0 & (1/B^{\frac{1}{2}})\sin B^{\frac{1}{2}}T \\ -A^{\frac{1}{2}}\sin A^{\frac{1}{2}}T & 0 & \cos A^{\frac{1}{2}}T & 0 \\ 0 & -B^{\frac{1}{2}}\sin B^{\frac{1}{2}}T & 0 & \cos B^{\frac{1}{2}}T \end{pmatrix} \tag{63}$$

and consequently

$$\text{trace}=2(\cos A^{\frac{1}{2}}T+\cos B^{\frac{1}{2}}T). \tag{64}$$

If λ_3, λ_4 are the two conjugate characteristic roots of a periodic orbit, we have the relations $\lambda_3+\lambda_4=\text{trace}-2$ and $\lambda_3\lambda_4=1$. Consequently, from Eq. (64) we obtain

$$\lambda_{3,4}=\cos A^{\frac{1}{2}}T+\cos B^{\frac{1}{2}}T-1 \pm i[(\cos A^{\frac{1}{2}}T+\cos B^{\frac{1}{2}}T)(2-\cos A^{\frac{1}{2}}T-\cos B^{\frac{1}{2}}T)]^{\frac{1}{2}}. \tag{65}$$

Let us consider now the periodic orbit which oscillates up and down along the y axis:

$$\bar{x}=\bar{X}=0,\quad \bar{y}=(\bar{Y}_0/B^{\frac{1}{2}})\sin B^{\frac{1}{2}}t,\quad \bar{Y}=\bar{Y}_0\cos B^{\frac{1}{2}}t. \tag{66}$$

The period of this orbit is equal to $T=2\pi/B^{\frac{1}{2}}$ and its two conjugate characteristic roots are found from Eq. (65) to be

$$\lambda_{3,4}=\cos(2\pi A^{\frac{1}{2}}/B^{\frac{1}{2}})\pm i\sin(2\pi A^{\frac{1}{2}}/B^{\frac{1}{2}}), \tag{67}$$

from which we obtain the characteristic exponents

$$\pm i\alpha=\pm i(2\pi/T)(A^{\frac{1}{2}}/B^{\frac{1}{2}}). \tag{68}$$

If in Eq. (63) we set $T=2\pi/B^{\frac{1}{2}}$ and take into account that the eigenvectors $f_3(0)\pm if_4(0)$ corresponding to the eigenvalues (67) are of the form given by the last two equations of (42), we find that in this case we have $a_2=b_2=0$ and $c=-a_3/(b_1A^{\frac{1}{2}})$. Hence, from Eq. (58) we obtain $\sigma=-A^{\frac{1}{2}}$ and using Eqs. (61) and (68), we obtain finally that the rotation number r is

$$r=-(A/B)^{\frac{1}{2}}\ (\text{mod } 1). \tag{69}$$

The value of r given by Eq. (61) is the maximum or minimum rotation number corresponding to the central periodic orbit, according to the value of $A-4B$. In the case $A=1.6$, $B=0.9$, the value of r is the maximum rotation number for each value of ϵ. When $\epsilon=0$ we find from Eq. (69) that $r=0.666\cdots$. For $\epsilon>0$ the maximum rotation number increases as ϵ increases up to the value $r=1$ for $\epsilon=4.305$. For larger ϵ the central periodic orbit becomes unstable. This is shown in Fig. 3, which gives the trace and the rotation number as a function of ϵ.

It is seen that the value $\epsilon=4.305$ is smaller than the escape perturbation $\epsilon_{esc}=4.6017$, which is marked in Fig. 3. This is the value of ϵ for which the curve of zero velocity

$$Ax^2+By^2-2\epsilon xy^2=2h$$

opens up and the moving point may go to infinity. Hence

$$\epsilon_{esc}=\tfrac{1}{2}B(A/2h)^{\frac{1}{2}}.$$

In other cases the central periodic orbit is stable even beyond the escape perturbation.

The values of r corresponding to the invariant curves very near the central periodic orbit, for each value of ϵ, are given by Eq. (34).

III. STABILITY OF THE ORBIT $y=0$

The equations of motion in the potential (2) are

$$dx/dt=X,\qquad dy/dt=Y,$$
$$dX/dt=-Ax+\epsilon y^2,\quad dY/dt=-By+2\epsilon xy, \tag{70}$$

and admit the periodic solution

$$x=x_0\cos A^{\frac{1}{2}}t+(X_0/A^{\frac{1}{2}})\sin A^{\frac{1}{2}}t,\quad y=0,$$
$$X=-A^{\frac{1}{2}}x_0\sin A^{\frac{1}{2}}t+X_0\cos A^{\frac{1}{2}}t,\quad Y=0 \tag{71}$$

with period $T=2\pi/A^{\frac{1}{2}}$. We note that this solution is independent of the value of ϵ. Its stability, however, depends on ϵ, as we show below.

The variational equations of the system (70) associated with the periodic solution (71) are

$$d\xi_1/dt=\xi_3,\qquad d\xi_2/dt=\xi_4,$$
$$d\xi_3/dt=-A\xi_1,\quad d\xi_4/dt=-B\xi_2+2\epsilon x\xi_2, \tag{72}$$

where x is given by the first equation (71). Without loss of generality we can set $x_0=0$; then

$$x=(2h/A)^{\frac{1}{2}}\sin A^{\frac{1}{2}}t. \tag{73}$$

The system (72) can be separated into two independent systems. The first, containing ξ_1 and ξ_3, is independent of ϵ and its general solution is

$$\xi_1=\xi_1(0)\cos A^{\frac{1}{2}}t+[\xi_3(0)/A^{\frac{1}{2}}]\sin A^{\frac{1}{2}}t,$$
$$\xi_3=-\xi_1(0)A^{\frac{1}{2}}\sin A^{\frac{1}{2}}t+\xi_3(0)\cos A^{\frac{1}{2}}t. \tag{74}$$

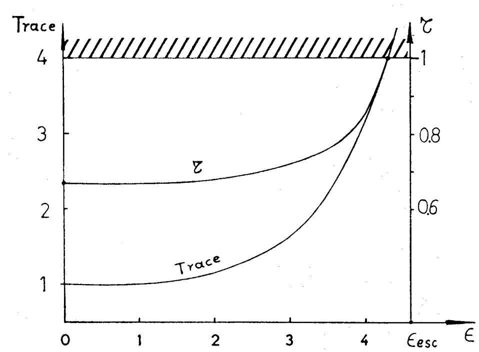

FIG. 3. The trace and the rotation number of the central periodic orbit corresponding to $A=1.6$, $\epsilon=0.9$, $h=0.00765$, as a function of ϵ. The unstable region is shaded.

The second part can be written in the form

$$\ddot{\xi}_2=-(B-2k\epsilon\sin A^{\frac{1}{2}}t)\xi_2, \qquad (75)$$

which is a Mathieu equation. Its solution can be obtained as a series in powers of ϵ, of the form

$$\xi_2=\xi_{20}+\epsilon\xi_{21}+\epsilon^2\xi_{22}+\cdots. \qquad (76)$$

If we substitute Eq. (76) into Eq. (75) and equate the terms of the same order, we find that the functions ξ_{2n} satisfy the systems

$$\dot{\xi}_{20}=\xi_{40}, \quad \dot{\xi}_{40}=-\xi_{20}, \qquad (77)$$

$$\begin{aligned}&\dot{\xi}_{2n}=\xi_{4n} \quad (n=1,2,\cdots),\\ &\dot{\xi}_{4n}=-B\xi_{2n}+2(2h/A)^{\frac{1}{2}}\sin A^{\frac{1}{2}}t\xi_{2,n-1}.\end{aligned} \qquad (78)$$

We note that the systems (78) are linear nonhomogeneous systems in ξ_{2n}, ξ_{4n}. We call $\Delta(t)$ the matrix

$$\Delta(t)=\begin{pmatrix}\cos B^{\frac{1}{2}}t & B^{-\frac{1}{2}}\sin B^{\frac{1}{2}}t\\ -B^{\frac{1}{2}}\sin B^{\frac{1}{2}}t & \cos B^{\frac{1}{2}}t\end{pmatrix}, \qquad (79)$$

whose columns are the solutions of the homogeneous part of the systems (78), corresponding to the initial conditions (1,0) and (0,1), respectively. Then a solution of the complete system (78) is

$$\xi_{2n}=\sum_{j=1}^{2}\alpha_{1j}\int_0^t\Delta_{2j}R_n dt, \quad \xi_{4n}=\sum_{j=1}^{2}\alpha_{2j}\int_0^t\Delta_{2j}R_n dt, \qquad (80)$$

where the α_{ij} are the elements of the matrix (79), the Δ_{ij} its minors and the R_n the nonhomogeneous part of Eqs. (78). Applying Eqs. (80) iteratively, we find that the function $\xi_2^{\mathrm{I}}(t)$, corresponding to the initial conditions (1,0) and the function $\xi_4^{\mathrm{II}}(t)$, corresponding to the initial conditions (0,1) are up to second-order terms in ϵ,

$$\begin{aligned}\xi_2^{\mathrm{I}}=\cos B^{\frac{1}{2}}t&-\frac{(2h)^{\frac{1}{2}}\epsilon}{AB^{\frac{1}{2}}(A-4B)}[B^{\frac{1}{2}}(A^{\frac{1}{2}}-2B^{\frac{1}{2}})\sin(A^{\frac{1}{2}}+B^{\frac{1}{2}})t+B^{\frac{1}{2}}(A^{\frac{1}{2}}+2B^{\frac{1}{2}})\sin(A^{\frac{1}{2}}-B^{\frac{1}{2}})t-2(A-2B)\sin B^{\frac{1}{2}}t]\\ &+\frac{h\epsilon^2}{2A(AB)^{\frac{1}{2}}(A-4B)}\Bigg[-4A^{\frac{1}{2}}t\sin B^{\frac{1}{2}}t-\frac{2A^{\frac{1}{2}}}{B^{\frac{1}{2}}}\cos B^{\frac{1}{2}}t-\frac{B^{\frac{1}{2}}(A^{\frac{1}{2}}-2B^{\frac{1}{2}})}{A^{\frac{1}{2}}(A^{\frac{1}{2}}+B^{\frac{1}{2}})}\cos(2A^{\frac{1}{2}}+B^{\frac{1}{2}})t-\frac{B^{\frac{1}{2}}(A^{\frac{1}{2}}+2B^{\frac{1}{2}})}{A^{\frac{1}{2}}(A^{\frac{1}{2}}-B^{\frac{1}{2}})}\cos(2A^{\frac{1}{2}}-B^{\frac{1}{2}})t\\ &+\frac{8(A-2B)}{A^{\frac{1}{2}}(A^{\frac{1}{2}}+2B^{\frac{1}{2}})}\cos(A^{\frac{1}{2}}+B^{\frac{1}{2}})t-\frac{8(A-2B)}{A^{\frac{1}{2}}(A^{\frac{1}{2}}-2B^{\frac{1}{2}})}\cos(A^{\frac{1}{2}}-B^{\frac{1}{2}})t-2D\cos B^{\frac{1}{2}}t\Bigg]. \end{aligned} \qquad (81)$$

and

$$\begin{aligned}\xi_4^{\mathrm{II}}=\cos B^{\frac{1}{2}}t&-\frac{(2h/B)^{\frac{1}{2}}\epsilon}{A(A-4B)}[(A^{\frac{1}{2}}-2B^{\frac{1}{2}})(A^{\frac{1}{2}}+B^{\frac{1}{2}})\sin(A^{\frac{1}{2}}+B^{\frac{1}{2}})t-(A^{\frac{1}{2}}+2B^{\frac{1}{2}})(A^{\frac{1}{2}}-B^{\frac{1}{2}})\sin(A^{\frac{1}{2}}-B^{\frac{1}{2}})t+4B\sin B^{\frac{1}{2}}t]\\ &+\frac{h\epsilon^2}{2A(AB)^{\frac{1}{2}}(A-4B)}\Bigg[-\frac{(A^{\frac{1}{2}}-2B^{\frac{1}{2}})(2A^{\frac{1}{2}}+B^{\frac{1}{2}})}{A^{\frac{1}{2}}(A^{\frac{1}{2}}+B^{\frac{1}{2}})}\cos(2A^{\frac{1}{2}}+B^{\frac{1}{2}})t+\frac{(A^{\frac{1}{2}}+2B^{\frac{1}{2}})(2A^{\frac{1}{2}}-B^{\frac{1}{2}})}{A^{\frac{1}{2}}(A^{\frac{1}{2}}-B^{\frac{1}{2}})}\cos(2A^{\frac{1}{2}}-B^{\frac{1}{2}})t\\ &-\frac{16B^{\frac{1}{2}}(A^{\frac{1}{2}}+B^{\frac{1}{2}})}{A^{\frac{1}{2}}(A^{\frac{1}{2}}+2B^{\frac{1}{2}})}\cos(A^{\frac{1}{2}}+B^{\frac{1}{2}})t-\frac{16B^{\frac{1}{2}}(A^{\frac{1}{2}}-B^{\frac{1}{2}})}{A^{\frac{1}{2}}(A^{\frac{1}{2}}-2B^{\frac{1}{2}})}\cos(A^{\frac{1}{2}}-B^{\frac{1}{2}})t-4A^{\frac{1}{2}}t\sin B^{\frac{1}{2}}t+\frac{4A^{\frac{1}{2}}}{B^{\frac{1}{2}}}\cos B^{\frac{1}{2}}t+2D'\cos B^{\frac{1}{2}}t\Bigg], \end{aligned} \qquad (82)$$

where D, D' are constants such that the second-order terms of ξ_2^{I}, ξ_4^{II} are equal to zero when $t=0$.

The stability of the orbit (73) is studied by calculating the trace of the monodromy matrix. The first diagonal element of the monodromy matrix of the system (72) is equal to $\xi_1(T)$, given by the first equation (74), for $\xi_1(0)=1$, $\xi_3(0)=0$, and the third diagonal element is equal to $\xi_3(T)$, given by the second equation (74), for $\xi_1(0)=0$, $\xi_3(0)=1$. The second diagonal element is $\xi^{\mathrm{I}}(T)$, given by Eq. (81) and the fourth is $\xi_4^{\mathrm{II}}(T)$, given by Eq. (82). Hence, finally, we obtain that the trace of the monodromy matrix of the system (72) is, up to second-order terms in ϵ,

$$\text{trace}=2+2\cos 2\pi\frac{B^{\frac{1}{2}}}{A^{\frac{1}{2}}}-\frac{8\pi h\epsilon^2}{A(AB)^{\frac{1}{2}}(A-4B)}\sin\left(2\pi\frac{B^{\frac{1}{2}}}{A^{\frac{1}{2}}}\right). \qquad (83)$$

The trace is between 0 and 4 if ϵ is small and the ratio $B^{\frac{1}{2}}/A^{\frac{1}{2}}$ is not near $\frac{1}{2}$ or 1. Therefore the orbit $y=0$ is stable for small ϵ, unless the ratio $A^{\frac{1}{2}}/B^{\frac{1}{2}}$ is near 2 or 1.

In the case $A\simeq 4B$ we write

$$4B=A+\epsilon k; \qquad (84)$$

then $2B^{\frac{1}{2}}/A^{\frac{1}{2}}=1+(\epsilon k/2A)+\cdots$; hence, to second-order approximation in ϵ,

$$\text{trace}=(\pi^2\epsilon^2/4A^2)[k^2-(32h/A)]. \qquad (85)$$

The orbit is unstable if

$$-4(2h/A)^{\frac{1}{2}}<k<4(2h/A)^{\frac{1}{2}}. \qquad (86)$$

This result is the same as that found in a different way by Contopoulos and Moutsoulas (1966). Similar results appear whenever $A\simeq B$.

If ϵ is large the orbit $y=0$ may become unstable. This is seen in Fig. 4. In the case $A=1.6$, $B=0.9$ we have from Eq. (83), for small ϵ,

$$\text{trace}=2-(\pi h\epsilon^2/0.48). \tag{87}$$

For $\epsilon=2$ and $h=0.00765$ we find trace$=1.80$ and for $\epsilon=4$ trace$=1.20$ while the real values are 1.80 and 1.24. Therefore the above formula gives roughly correct results even for fairly large values of ϵ.

The value $\epsilon_{esc}=4.6017$ is marked in Fig. 4. We notice that the orbit $y=0$ remains stable even beyond the escape perturbation. Therefore orbits near the periodic orbit $y=0$ do not go to infinity, although the curve of zero velocity is open.

The rotation number for an orbit near $y=0$ can be found as follows. When such an orbit crosses for the nth time the x axis going upward we have $\xi_2^{\mathrm{I}}=0$, $\dot{\xi}_2^{\mathrm{I}}>0$, hence

$$B^{\frac{1}{2}}T_n=\tfrac{3}{2}\pi+2n\pi+\epsilon q_1+\epsilon^2 q_2, \tag{88}$$

where q_1 and q_2 are to be found from Eq. (81). At this instant we have

$$\bar{x}=A^{\frac{1}{2}}x=[(2h)^{\frac{1}{2}}+\xi_3(0)]\sin A^{\frac{1}{2}}T_n+\xi_1(0)A^{\frac{1}{2}}\cos A^{\frac{1}{2}}T_n,$$

and

$$X=[(2h)^{\frac{1}{2}}+\xi_3(0)]\cos A^{\frac{1}{2}}T_n-\xi_1(0)A^{\frac{1}{2}}\sin A^{\frac{1}{2}}T_n; \tag{89}$$

hence the angle between the origin and the point $(\bar{x},X)$ is

$$\theta_n=\varphi-A^{\frac{1}{2}}T_n, \tag{90}$$

where

$$\tan\varphi=[(2h)^{\frac{1}{2}}+\xi_3(0)]/[\xi_1(0)A^{\frac{1}{2}}]. \tag{91}$$

Thus the rotation number is

$$r=\lim\frac{\theta_n}{2\pi n}=-\frac{A^{\frac{1}{2}}}{2\pi nB^{\frac{1}{2}}}\left(\frac{3\pi}{2}+2n\pi+\epsilon q_1+\epsilon^2 q_2\right)(\mathrm{mod}\ 1). \tag{92}$$

It is easily seen from Eq. (81) that q_1 contains n only in trigonometric terms, while the part of q_2 which is proportional to n is provided by the secular term of order ϵ^2, i.e., it is contained in

$$q_2=\frac{2h\epsilon^2T_n\sin B^{\frac{1}{2}}T_n}{AB^{\frac{1}{2}}(A-4B)}+\cdots=\frac{2h\epsilon^2(2\pi n)}{AB(A-4B)}+\cdots. \tag{93}$$

Thus, when $n\to\infty$, Eq. (92) gives

$$r=-(A^{\frac{1}{2}}/B^{\frac{1}{2}})\{1-[2\epsilon^2h/AB(A-4B)]\},$$

and this value is the same as that given by Eq. (35).

IV. CONCLUSIONS AND APPLICATIONS

(1) We have noticed that islands and tube orbits appear only in nonseparable dynamical systems. In the general case of nonintegrable systems an infinite number of sets of islands appears. These can be found

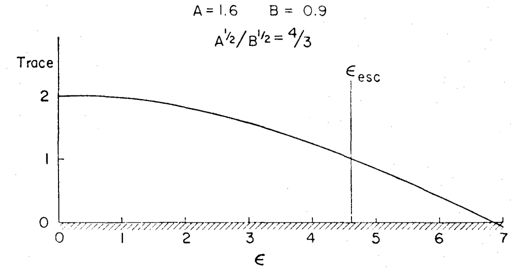

FIG. 4. The trace of the monodromy matrix corresponding to the orbit $y=0$, for $A=1.6$, $B=0.9$,, $h=0.00765$, as a function of ϵ. The unstable region is shaded.

approximately by means of the "third" integral, which takes a different form for each set of islands.

A practical application of the above considerations refers to the models of our Galaxy. Most models of our Galaxy are nonseparable and nonintegrable; thus the appearance of islands and tube orbits of different orders (Ollongren 1965) is justified. On the other hand, there are also a few separable models (e.g., Hori 1962). In these models no tube orbits appear at all. However, as we have noticed, separable systems are exceptional; therefore it is quite probable that tube orbits do exist in our Galaxy.

In order to find the positions and importance of the real tube orbits of our Galaxy (i.e., for how large a set of initial conditions we have tube orbits), a comparative study of various models of the Galaxy is necessary.

(2) In the case of a nonintegrable system which is near an integrable one the invariant curves on a surface of section are, in general, well defined; they are either regular invariant curves, or islands. On each invariant curve one can define a rotation number r. Then a rotation curve gives r as a function of the distance of each curve from the center. The rotation curve is a smooth function in integrable cases, but it has straight plateaus and small discontinuities at every rational number r in nonintegrable cases.

The form of the rotation curve can be found approximately by means of the "third" integral. The maximum r corresponds to the central periodic orbit, (which is near the y axis), and the minimum r to the orbit $y=0$ (or a nearby orbit), or vice versa. Thus we have the range of variation of r and consequently the kinds of islands and tube orbits that should be expected in every case.

(3) The maximum (or minimum) rotation number can be found accurately by means of the characteristic exponents of the periodic orbit near the y axis (central periodic orbit). We have calculated the characteristic exponents of this orbit and of the periodic orbit $y=0$ for various values of the perturbation ϵ.

In one case the central periodic orbit becomes unstable even before ϵ reaches the value of the escape perturbation ϵ_{esc}, while the orbit $y=0$ is stable well beyond it. In other cases the central periodic orbit is stable beyond $\epsilon=\epsilon_{esc}$. Therefore the stability of a periodic orbit is not directly related to the escape perturbation. It is related to a "third" integral, which exists in a certain region near the periodic orbit, sometimes even beyond the escape perturbation. Therefore one may find in our Galaxy stars with velocity greater than the velocity of escape, which do not escape, because they happen to be near a stable periodic orbit.

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